



SOLUTION OF SOME NON-LOCAL PROBLEM FOR MIXED EQUATION WITH SECOND ORDER

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It is considered hyperbolic equation

$$u_{xx} - u_{yy} = 0, \quad y < 0. \quad (1)$$

in the area which is bounded by characteristics

$$x + y = 0, \quad x - y = 1 \quad \text{and} \quad y = 0 \quad (2)$$

Theorem 1. The hyperbolic equation “(1)” with characteristics “(2)” can be solved with non-local boundary conditions

$$\begin{cases} u(x; 0) = \tau(x) \\ \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} u\left(\frac{x+2i}{2n}, -\frac{x+2i}{2n}\right) + \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} u\left(\frac{-x+2n-2i-1}{2n}, -\frac{x+2i+1}{2n}\right) = \phi(x) \end{cases} \quad (3)$$

where $\tau(x)$, $\phi(x)$ are twice differentiable functions, $\lfloor x \rfloor$ is greatest integer number that not exceed real number x .

Keywords: Elliptic-Hyperbolic, Parabolic-Hyperbolic, non-local problems for mixed equation, micro-periodic functional equations.

AMS Subject Classification: 35L51, 35M30, 35M32

1. INTRODUCTION

Consider the hyperbolic equation “(1)”.

Our main aim is to find regular solution of problem “(1)” with characteristics “(2)” and non-local boundary conditions “(3)” in explicit formula.

On this work we consider special cases of work [4] where shown the existence of solution to problem (1) but explicit solution are not shown.

To obtain the results we use several substitutions and methods which can be used to find the solutions of functional equation (see, for example, [4], [5], [1], [2], [3]).

2. GENERAL OBTAINED RESULTS

As we know the general solution of "(1)" is in form like below

$$u(x, y) = f(x + y) + F(x - y) \quad (4)$$

where $f(x), F(x)$ are twice differentiable functions.

Using boundary condition "(19)" we have :

$$f(x) = \tau(x) - F(x) \quad (5)$$

therefore by using "(4)" :

$$u\left(\frac{x+2i}{2n}; -\frac{x+2i}{2n}\right) = f(0) + F\left(\frac{x+2i}{n}\right),$$

$$u\left(\frac{-x+2n-2i-1}{2n}; -\frac{x+2i+1}{2n}\right) = f\left(\frac{-x+n-2i-1}{n}\right) + F(1)$$

using non-local boundary condition "(19)" we have:

$$\phi(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F\left(\frac{x+2i}{n}\right) + \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} f\left(\frac{-x+n-2i-1}{n}\right) + \lfloor \frac{n+1}{2} \rfloor f(0) + \lfloor \frac{n}{2} \rfloor F(1) \quad (6)$$

then by substituting "(5)" to the "(6)" we get:

$$\phi(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F\left(\frac{x+2i}{n}\right) + \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \left(\tau\left(\frac{-x+n-2i-1}{n}\right) - F\left(\frac{-x+n-2i-1}{n}\right) \right) + \lfloor \frac{n+1}{2} \rfloor f(0) + \lfloor \frac{n}{2} \rfloor F(1)$$

By putting

$$\mu(x) = \phi(x) - \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \tau\left(\frac{-x+n-2i-1}{n}\right) - \lfloor \frac{n+1}{2} \rfloor f(0) - \lfloor \frac{n}{2} \rfloor F(1)$$

and using the last equation we get following functional equation

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F\left(\frac{x+2i}{n}\right) - \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} F\left(\frac{-x+n-2i-1}{n}\right) = \mu(x). \quad (7)$$

Particular cases of equation "(7)" for $n = 1, 2, 3, 4$ were considered on works [1], [2], [3]. Now let's provide exact solution to the equation "(7)" for all natural numbers n .

The Case $n = 2k + 2$

In this case we solve functional equation "(7)" for even n and our equation will be in following form:

$$\sum_{i=0}^k F\left(\frac{x+2i}{2k+2}\right) - \sum_{i=0}^k F\left(\frac{2k+1}{2k+2} - \frac{x+2i}{2k+2}\right) = \mu(x). \quad (8)$$

Then by substituting

$$G(x) = F\left(\frac{x}{2k+2}\right) - F\left(\frac{2k+1}{2k+2} - \frac{x}{2k+2}\right) \quad (9)$$

and get the following equation

$$G(x) + G\left(x + \frac{1}{k+1}\right) + G\left(x + \frac{2}{k+1}\right) + \dots + G\left(x + \frac{k}{k+1}\right) = \mu(x) \quad (10)$$

Putting $x = y + \frac{1}{k+1}$ therefore

$$G\left(y + \frac{1}{k+1}\right) + G\left(y + \frac{2}{k+1}\right) + \dots + G\left(y + \frac{k}{k+1}\right) + G(y+1) = \mu\left(y + \frac{1}{k+1}\right).$$

Substitute $y = x$ and subtracting the last equation from "(10)" we get the following functional equation

$$G(x) - G(x + 1) = \mu(x) - \mu\left(x + \frac{1}{k + 1}\right).$$

The last equation is considered on [5] as microperiodic functional equation and has solution like below:

$$G(x) = H_0(x) + A\cos(2\pi x).$$

where $H_0(x)$ is particular solution to the last functional equation and A is some real number.

Put the obtained result to the equation "(9)" we get the following general solution (see [5])

$$F(x) = G_0(x) + B\sin\left(\frac{2k + 2}{2k + 1}\pi x\right),$$

where $G_0(x)$ is particular solution to the last functional equation and B is some real number.

The Case $n = 2k + 1$

In this case we solve functional equation "(7)" for odd n and our equation will be in following form:

$$\sum_{i=0}^k F\left(\frac{x + 2i}{2k + 1}\right) - \sum_{i=0}^{k-1} F\left(\frac{2k - 2i}{2k + 1} - \frac{x}{2k + 1}\right) = \mu(x). \tag{11}$$

Let $H(x + y) = F(x + y) - F(x - y)$ then substitute, $x = \frac{x}{2k + 1}$ and $y = \frac{2k - 2i}{2k + 1}$ to "(11)" we have:

$$H\left(\frac{x}{2k + 1} + \frac{2k - 2i}{2k + 1}\right) = F\left(\frac{x}{2k + 1} + \frac{2k - 2i}{2k + 1}\right) - F\left(\frac{2k - 2i}{2k + 1} - \frac{x}{2k + 1}\right) \tag{12}$$

If we substitute $x = y = \frac{x}{2(2k + 1)}$ to "(11)" we have:

$$H\left(\frac{x}{2k + 1}\right) = F\left(\frac{x}{2k + 1}\right) - F(0) \tag{13}$$

By using "(12)" and "(13)" in "(11)" after simplifying we get similar to "(10)":

$$H\left(\frac{x}{2k + 1}\right) + H\left(\frac{x + 2}{2k + 1}\right) + H\left(\frac{x + 4}{2k + 1}\right) + \dots + H\left(\frac{x + 2k}{2k + 1}\right) = \mu(x)$$

Above functional equation has following general solution (see [5])

$$H(x) = H_0(x) + A\sin\left(\frac{2k + 1}{2k}\pi x\right).$$

Finally we get the general solution for the functional equation "(7)". Substitute obtained solution depending on parity of natural number n into the equation "(5)" we will get function $f(x)$. Therefore by substituting $f(x)$ and $F(x)$ to the equation "(4)" we get explicit solution of problem "(1)".

Remark: If we consider the hyperbolic equation

$$u_{xx} - u_{yy} = P(x; y), \quad y < 0$$

with characteristics "(2)" and non-local boundary conditions "(19)" can be solved with the similar way.

3. GENERALIZATION FOR MIXED EQUATIONS WITH SECOND ORDER

Let us consider two following cases:

Case Lavrentiev-Bitsadze

Let's consider the Lavrentiev-Bitsadze equation

$$u_{xx} + \text{sign}(y)u_{yy} = 0, \quad (14)$$

in the area which is bounded by characteristics

$$x + y = 0, \quad x - y = 1 \quad \text{and} \quad \text{half-strip } \sigma(x; y) \quad (15)$$

Theorem 2. The Lavrentiev-Bitsadze equation “(14)” with characteristics “(15)” can be solved with non-local boundary conditions

$$\begin{cases} u(x; y)|_{\sigma} = \psi(s) \\ \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} u\left(\frac{x+2i}{2n}; -\frac{x+2i}{2n}\right) + \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} u\left(\frac{-x+2n-2i-1}{2n}; -\frac{x+2i+1}{2n}\right) = \phi(x) \end{cases} \quad (16)$$

where $\psi(s)$ is continuous function on half-strip $\sigma(x; y)$ and $\phi(x)$ is twice differentiable function, $\lfloor x \rfloor$ is greatest integer number that not exceed real number x .

Proof of theorem 2. The proof of the theorem can be straightly derived from the combination of *Theorem 1* and [2]

Case Hyperbolic-Parabolic

Let's consider the Hyperbolic-Parabolic equation

$$\begin{cases} u_{xx} + u_y = 0, & y > 0 \\ u_{xx} - u_{yy} = 0 & y < 0 \end{cases} \quad (17)$$

in the area which is bounded by characteristics

$$x = 0, x = 1, y = x - 1, \text{ and, } x + y = 0 \quad (18)$$

Theorem 3. The Hyperbolic-Parabolic equation “(17)” with characteristics “(18)” can be solved with non-local boundary conditions

$$\begin{cases} u(0; y) = \phi_1(x), & u(1; y) = \phi_2(x) \\ \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} u\left(\frac{x+2i}{2n}; -\frac{x+2i}{2n}\right) + \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} u\left(\frac{-x+2n-2i-1}{2n}; -\frac{x+2i+1}{2n}\right) = \phi(x) \end{cases} \quad (19)$$

where $\phi_1(x), \phi_2(x)$ is continuous functions on $[0; 1]$ and $\phi(x)$ is twice differentiable function, $\lfloor x \rfloor$ is greatest integer number that not exceed real number x .

Proof of theorem 3. The proof of the theorem can be straightly derived from the combination of *Theorem 1* and [3]

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